Necessary and Sufficient Conditions for Normal Goods

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1. Introduction

The usefulness of certain basic concepts in economic analysis is indisputable. Two such concepts are elasticity and the marginal rate of substitution (MRS). It is somewhat surprising, then, that when the topic at hand is the theory of consumer demand, the elasticities of the MRS are seldom highlighted or exploited, either in instruction or in research. The primary purpose of this paper is to demonstrate the usefulness of the “MRS elasticities” in characterizing a consumer’s response to changes in his or her income. Somewhat more narrowly, our main focus is on the usefulness of the MRS–elasticity approach in characterizing goods as normal or inferior.

The traditional comparative–statics approach to the consumer’s problem is to formulate a matrix system in which the elements of the coefficient matrix are (i) second–order partial derivatives of the utility function and (ii) prices. It can be argued that this approach is problematic in that it produces expressions for the partial derivatives of the demand functions that involve the second–order partial derivatives of the utility function. These second–order partial derivatives are not invariant with respect to increasing monotonic transformations of the utility function; that is, second–order partials do not display what we will call the invariance property.

We know, of course, that the consumer’s demand functions and their partial derivatives do display the invariance property. Consider, for example, a certain consumer with a particular strictly–quasiconcave utility function, \( u(x_1, x_2, \ldots x_n) \). Given initial prices and income, suppose the consumer demands five units of good 1. Now, the consumer experiences an increase in income of a certain amount, after which he demands six units of good 1. If, instead of \( u \), the consumer’s utility function were \( v = \phi(u(x_1, x_2, \ldots x_n)) \), with \( \phi'(u) > 0 \), then his actions would be the same as if his utility function were \( u \).

The inclusion of the second–order partials in the expression for the partial derivative of, say, good 1 with respect to income, however, introduces a “black box;” the consumer’s response displays, in every respect, the invariance property, while the individual determinants of that response do not display that property.

We propose an MRS–elasticity approach to the consumer’s problem as an alternative approach to the standard one. The approach is in step with previous and ongoing efforts to characterize consumer behavior, to the extent possible, in terms of the properties of the
utility function. Continuing a line of inquiry featuring contributions by Chipman (1977), Leroux (1987), Alarie, et al. (1990), and others, Bilancini and Boncinelli (2010) provide a review of utility–based analyses of normal goods, and they introduce, for the n–good case, a necessary and sufficient condition for normality of good \( i \) \( (i = 1, 2, \ldots, n) \). That condition is checked by calculation of certain minors of the bordered Hessian matrix. While they do not mention the concept of an MRS elasticity, Bilancini and Boncinelli (2010, p. 13) do observe that their condition is best interpreted by considering “the problem of the consumer who faces a marginal increase in income as the problem of finding, in the consumption space, the direction along which pairwise marginal rates of substitution do not change.”

2. Alternative Methods for Comparative–Statics Analysis

The consumer’s problem is to maximize the strictly–quasiconcave, twice continuously–differentiable utility function, \( u(x_1, x_2, \ldots, x_n) \), subject to the budget constraint, \( \sum_{i=1}^{n} p_i x_i = I \), where \( p_i \) is the price of good \( i \), \( I \) is the consumer’s income, and the \( p_i \) and \( I \) are exogenous. We will use the usual notation for marginal utilities (MUs) and for the partial derivatives of the MUs: \( u_i \equiv MU_i \equiv \partial u(x_1, x_2, \ldots, x_n)/\partial x_i \) and \( u_{ij} \equiv \partial u_i(x_1, x_2, \ldots, x_n)/\partial x_j \). We assume that \( MU_i > 0 \) for all \( i \).

One approach to a comparative–statics analysis of the problem is to use, directly, the first–order conditions as they apply to each of the \( n \) goods:

\[
u_i - \lambda p_i = 0 \quad i = 1, 2, \ldots, n. \tag{1}\]

In (1), \( \lambda \) is the Lagrange multiplier. With this approach, we totally differentiate (1) and the budget constraint, yielding \( n + 1 \) equations in \( n + 1 \) unknowns; the unknowns are the \( dx_i \) and \( d\lambda \). When put into matrix form, the coefficient matrix is the bordered Hessian matrix, and the demand functions, along with the partial derivatives of \( \lambda \) with respect to the \( p_i \), are determined by these \( n + 1 \) equations in the usual way.

An alternative approach is to start with the familiar implication of (1): At the optimum, each marginal rate of substitution (MRS) must equal a certain ratio of prices. Denote the MRS between goods \( i \) and \( j \) by \( MRS_{ij} \equiv u_i/u_j \). From (1) we have, for goods \( i \) and \( j \),

\[
p_j MRS_{ij} = p_i \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2}\]

In (2) we have \( n – 1 \) linearly independent equations in \( n \) unknowns; the unknowns are the
dx_i. Total differentiation of (2) and a bit of rearrangement yields

\[ p_j \left( \frac{\partial (MRS_{ij})}{\partial x_1} \cdot dx_1 + \frac{\partial (MRS_{ij})}{\partial x_2} \cdot dx_2 + \ldots + \frac{\partial (MRS_{ij})}{\partial x_n} \cdot dx_n \right) = dp_i - MRS_{ij} \cdot dp_j \quad (3) \]

Throughout the remainder of the paper, given a small change in any variable \( z \), we will use \( \delta z \) to represent the corresponding percentage change in \( z \); that is, \( \delta z \equiv dz/z \). Use \( \epsilon_{ijk} \) to denote the elasticity of \( MRS_{ij} \) with respect to the quantity of good \( k \):

\[ \epsilon_{ijk} \equiv \frac{\delta MRS_{ij}}{\delta x_k} = \frac{x_k}{MRS_{ij}} \cdot \delta x_k \] \[ i, j, k = 1, 2, \ldots, n, \quad i \neq j. \quad (4) \]

(Good \( k \) may or may not be the same as either good \( i \) or good \( j \).) We will refer to the \( \epsilon_{ijk} \) as the \textit{MRS elasticities}. On the left–hand side of (3), multiply and divide the first term by \( MRS_{ij}/x_1 \), multiply and divide the second term by \( MRS_{ij}/x_2 \), and so on. Doing so and simplifying, we obtain the following result:

\[ \epsilon_{ij1} \delta x_1 + \epsilon_{ij2} \delta x_2 + \cdots + \epsilon_{ijn} \delta x_n = \delta p_i - \delta p_j. \quad (5) \]

Let \( \nu_i \equiv p_i x_i/I \) denote the expenditure–share on good \( i \). Totally differentiating the budget constraint and rearranging, we obtain

\[ \nu_1 \delta x_1 + \nu_2 \delta x_2 + \cdots + \nu_n \delta x_n = \delta I - \left( \nu_1 \delta p_1 + \nu_2 \delta p_2 + \cdots + \nu_n \delta p_n \right). \quad (6) \]

Expressing (5) and (6) in matrix form, we have

\[
\begin{bmatrix}
\epsilon_{121} & \epsilon_{122} & \cdots & \epsilon_{12n} \\
\epsilon_{231} & \epsilon_{232} & \cdots & \epsilon_{23n} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{n-1,n1} & \epsilon_{n-1,n2} & \cdots & \epsilon_{n-1,nn} \\
\nu_1 & \nu_2 & \cdots & \nu_n \\
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\vdots \\
\delta x_{n-1} \\
\delta x_n \\
\end{bmatrix}
=
\begin{bmatrix}
\delta p_1 - \delta p_2 \\
\delta p_2 - \delta p_3 \\
\vdots \\
\delta p_{n-1} - \delta p_n \\
\delta I - \sum_{i=1}^{n} \nu_i \delta p_i \\
\end{bmatrix}
\] \quad (7)

In (7) we have \( n \) linearly independent equations in \( n \) unknowns; the unknowns are the \( \delta x_i \).

The primary advantage of the approach represented by (7) relative to the bordered–Hessian approach is that, in (7), every element of the coefficient matrix is invariant to
(increasing) monotonic transformations of the utility function. (The MRSs are invariant to such transformations, so the MRS elasticities are likewise invariant.) Thus, our approach is advantageous in that it demystifies the “black box” that results from focusing on the individual second–order partial derivatives of the utility function, which do not, in general, display the invariance property.

Moreover, there are at least two secondary advantages of our approach. One is that, in solving (7), we directly obtain expressions for the elasticities of demand in a neighborhood of the consumer’s optimum. For example, if we hold the prices constant, we can solve (7) directly for the income elasticities, \( \delta x_i(x_1, x_2, \ldots, x_n)/\delta I \). If we are inclined to recover the partial derivative of \( x_i \) with respect to \( I \), then we can do so simply by taking \( (\delta x_i/\delta I) \cdot (I/x_i) \).

Finally, another advantage of the approach is that the elimination of the Lagrange multiplier reduces the number of independent equations (which is the same as the number of unknowns) from \( n + 1 \) to \( n \).

In order to determine the signs at least some of the demand elasticities, we need to assume something about the signs of the MRS elasticities. Our primary assumption of this type is as follows:

\[ \text{(A1)} \quad \text{For any } i, j, \text{ we assume } \epsilon_{ijj} > 0. \]

Assumption (A1) is the assumption that \( \text{MRS}_{ij} \equiv u_i/u_j \) increases with the quantity of good \( j \). Suppose, in accordance with the principle of diminishing marginal utility, \( u_{jj} < 0 \). Then the second–order cross–partial derivative \( u_{ij} \) would have to be negative and sufficiently large in absolute value in order for (A1) not to hold. That is, if \( u_{ij} < 0 \), then goods \( i \) and \( j \) would have to be relatively–strong Auspitz–Lieben–Edgeworth–Pareto (ALEP) substitutes in order for (A1) not to hold.

It is important to note that (A1) carries with it the assumption that \( \epsilon_{jij} < 0 \) for any \( i, j \). (\( \text{MRS}_{ji} \equiv u_j/u_i \) decreases with the quantity of good \( j \).) This follows simply from the fact that \( \text{MRS}_{ji} \) is the reciprocal of \( \text{MRS}_{ij} \). Specifically, \( \epsilon_{jij} = -\epsilon_{ijj} \).

2.1 The Case of Two Goods

For the case of two goods, the only MRS is \( \text{MRS}_{12} \), and (7) becomes
The second-order sufficient condition (SOSC) for a constrained maximum is that the determinant of the coefficient matrix is negative, that is, \( \Delta_2 \equiv \nu_2 \epsilon_{121} - \nu_1 \epsilon_{122} < 0 \). The income elasticities are

\[
\frac{\delta_{x_1}}{\delta_I} = -\frac{\epsilon_{122}}{\Delta_2}, \quad \frac{\delta_{x_2}}{\delta_I} = \frac{\epsilon_{121}}{\Delta_2},
\]

(9)

It follows from assumption (A1) and (9) that both goods are normal. We know, then, that the own–price elasticities are negative. The own–price elasticities are

\[
\frac{\delta_{x_1}}{\delta p_1} = \frac{\nu_2 + \nu_1 \epsilon_{122}}{\Delta_2}, \quad \frac{\delta_{x_2}}{\delta p_2} = \frac{\nu_1 - \nu_2 \epsilon_{121}}{\Delta_2}.
\]

(10)

Note that the necessary and sufficient condition for good \( i \) (\( i = 1, 2 \)) to be a law–of–demand good involves the expenditure shares as well as the MRS elasticities. Finally, the cross–price elasticities are

\[
\frac{\delta_{x_1}}{\delta p_2} = \frac{\nu_2 \cdot (\epsilon_{122} - 1)}{\Delta_2}, \quad \frac{\delta_{x_2}}{\delta p_1} = -\frac{\nu_1 \cdot (\epsilon_{121} + 1)}{\Delta_2}.
\]

(11)

One implication of (11) is that the cross–price elasticities can have different signs. This points up the ambiguity associated with the notion of gross complementarity/gross substitutability.

2.2 The Case of Three Goods

For the case of three goods, there are two independent MRSs. We chose to work with MRS_{12} and MRS_{23}. For this case, (7) becomes

\[
\begin{bmatrix}
\epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\
\epsilon_{231} & \epsilon_{232} & \epsilon_{233} \\
\nu_1 & \nu_2 & \nu_3
\end{bmatrix}
\begin{bmatrix}
\delta_{x_1} \\
\delta_{x_2} \\
\delta_{x_3}
\end{bmatrix}
= 
\begin{bmatrix}
\delta_{p_1} - \delta_{p_2} \\
\delta_{p_2} - \delta_{p_3} \\
\delta_I - \nu_1 \delta_{p_1} - \nu_2 \delta_{p_2} - \nu_3 \delta_{p_3}
\end{bmatrix}.
\]

(12)
The SOSC is that the determinant of the coefficient matrix is positive. Now, it is relatively easy to verify that, in general, \( \epsilon_{ijk} = \epsilon_{ikk} - \epsilon_{jkk} \). In particular, \( \epsilon_{123} = \epsilon_{133} - \epsilon_{233} \) and \( \epsilon_{231} = \epsilon_{211} - \epsilon_{311} \). Also, recall that \( \epsilon_{iji} = -\epsilon_{jii} \). In particular, \( \epsilon_{121} = -\epsilon_{211} \) and \( \epsilon_{232} = -\epsilon_{322} \). Using these facts, we can express the SOSC as

\[
\Delta_3 \equiv \nu_1(\epsilon_{122}\epsilon_{233} + \epsilon_{133}\epsilon_{322} - \epsilon_{233}\epsilon_{322}) + \nu_2(\epsilon_{133}\epsilon_{211} + \epsilon_{233}\epsilon_{311} - \epsilon_{133}\epsilon_{311})
\]

\[
+ \nu_3(\epsilon_{122}\epsilon_{311} + \epsilon_{211}\epsilon_{322} - \epsilon_{122}\epsilon_{211}) > 0.
\]  

(13)

The numerator of the income elasticity \( \frac{\delta x_i}{\delta I} \) is the term by which \( \nu_i \) is multiplied in (13). It is helpful to express each of those three terms in two different ways; for example, the term by which \( \nu_1 \) is multiplied in (13), \( \epsilon_{122}\epsilon_{233} + \epsilon_{133}\epsilon_{322} - \epsilon_{233}\epsilon_{322} \), can also be written as \( \epsilon_{322}(\epsilon_{133} - \epsilon_{233}) + \epsilon_{122}\epsilon_{233} \) or as \( \epsilon_{233}(\epsilon_{122} - \epsilon_{322}) + \epsilon_{133}\epsilon_{322} \). Accordingly, we can express each of the three income elasticities in two different ways:

\[
\frac{\delta x_1}{\delta I} = \frac{\epsilon_{322}(\epsilon_{133} - \epsilon_{233}) + \epsilon_{122}\epsilon_{233}}{\Delta_3}
= \frac{\epsilon_{233}(\epsilon_{122} - \epsilon_{322}) + \epsilon_{133}\epsilon_{322}}{\Delta_3}
\]  

(14)

\[
\frac{\delta x_2}{\delta I} = \frac{\epsilon_{133}(\epsilon_{211} - \epsilon_{311}) + \epsilon_{233}\epsilon_{311}}{\Delta_3}
= \frac{\epsilon_{311}(\epsilon_{233} - \epsilon_{133}) + \epsilon_{133}\epsilon_{322}}{\Delta_3}
\]  

(15)

\[
\frac{\delta x_3}{\delta I} = \frac{\epsilon_{122}(\epsilon_{311} - \epsilon_{211}) + \epsilon_{211}\epsilon_{322}}{\Delta_3}
= \frac{\epsilon_{211}(\epsilon_{322} - \epsilon_{122}) + \epsilon_{122}\epsilon_{311}}{\Delta_3}
\]  

(16)

Given assumption (A1) and the two expressions in (14), good 1 is normal \( (\frac{\delta x_1}{\delta I} > 0) \) if either \( \epsilon_{133} - \epsilon_{233} > 0 \) or if \( \epsilon_{122} - \epsilon_{322} > 0 \). If the first of these two inequalities holds, then an increase in \( x_3 \), holding \( x_1 \) and \( x_2 \) fixed, results in an increase in MRS_{13} that exceeds the increase in MRS_{23}. If the second inequality in (14) holds, then an increase in \( x_2 \), holding
\(x_1\) and \(x_3\) fixed, results in an increase in \(MRS_{12}\) that exceeds the increase in \(MRS_{32}\). The foregoing suggests a definition.

**Definition.** Choose any two goods \(i\) and \(j\). Good \(i\) is *MRS dominant* with respect to good \(j\) if, holding the quantities of all goods except \(j\) constant, there exists a good \(k\) such that an increase in good \(j\) increases \(MRS_{ij}\) by more than it increases \(MRS_{jk}\).

Based on this definition and on (14) – (16), we have a proposition.

**Proposition.** Assume (A1) holds. Then, for the \(n = 3\) case:

(a) Any one of the three goods is normal if that good is MRS dominant with respect to at least one of the other two goods.

(b) All three goods are normal if each good is MRS dominant with respect to at least one of the other two goods.

For the \(n = 3\) case, the interested reader may wish to verify that at most one good can be inferior. (Assume, for the sake of argument, that any two of the three goods are inferior. The result is a contradiction.)

### 2.3 Strict Concavity and ALEP Complementarity

For the general case of \(n\) goods, Chipman (1977) showed that if the utility function is strictly concave and if \(u_{ij} \geq 0\) (goods \(i\) and \(j\) are *weak ALEP complements*) for all \(i, j\), then all \(n\) goods are normal. For the two- and three-good cases, we will relate our results to Chipman’s.

For any \(i, j\) and allowing for \(i = j\), use \(\eta_{ij}\) to denote the elasticity of the marginal utility of good \(i\) with respect to the quantity of good \(j\):

\[
\eta_{ij} \equiv \frac{\partial u_i}{\partial x_j} \cdot \frac{x_j}{u_i} \equiv \frac{x_j u_{ij}}{u_i}.
\]

(17)

It is straightforward to verify that \(\epsilon_{ijk} = \eta_{ik} - \eta_{jk}\). For \(i = j\) we have \(\epsilon_{ijk} = \eta_{ik} - \eta_{kk}\) and for \(i = k\) we have \(\epsilon_{ijk} = \eta_{ij} - \eta_{ji}\). Chipman’s assumptions — \(u_{ij} \geq 0\) and strict concavity of \(u(\cdot)\) — imply that our assumption (A1) necessarily holds and that \(\eta_{ij} \geq 0\). As we showed
for the two–good case, (A1) ensures that both goods are normal, irrespective of whether \( u(\cdot) \) is strictly concave. For the three-good case, we can use (17) to rewrite (14) as follows:

\[
\frac{\delta x_1}{\delta I} = \frac{(\eta_{22}\eta_{33} - \eta_{23}\eta_{32}) + \eta_{12}\epsilon_{233} + \eta_{13}\epsilon_{322} \Delta_3}{\Delta_3}.
\] (18)

In the numerator on the right–hand side of (18), if \( u(\cdot) \) is strictly concave, then the term within parentheses is positive. If the assumption of weak ALEP complementarity holds, then the other terms in the numerator are nonnegative. Then good 1 is normal under Chipman’s assumptions. From (15) and (16) we can derive expressions similar to (18) that ensure normality of goods 2 and 3 under the same assumptions. Without strict concavity, (A1) by itself is not a sufficient condition for normality of all goods, and we have to use the MRS–dominance condition or something similar to ensure that all goods are normal.

3. Concluding Remarks

We have derived, for the cases of two and three goods, expressions for the income elasticities of demand in which all of the relevant determinants of those elasticities are invariant to monotonic transformations of the utility function. This stands in contrast to the results of the standard comparative–statics method in which second–order partial derivatives of the utility function — which do not display the invariance property — show up in the demand functions as determinants of the income elasticities. Our method, therefore, helps to clarify the determinants of the income elasticities, without direct regard to the signs or magnitudes of the second–order partial derivatives.

The analysis should be, in some way, extendable to the general case of \( n \) goods. The idea of MRS dominance that we introduced here may continue to play a key role in the more general analysis.
References


